LINKED ALTERNATING FORMS AND LINKED SYMPLECTIC GRASSMANNIANS

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Abstract. Motivated by applications to higher-rank Brill-Noether theory and the Bertram-Feinberg-Mukai conjecture, we introduce the concepts of linked alternating and linked symplectic forms on a chain of vector bundles, and show that the linked symplectic Grassmannians parametrizing chains of subbundles isotropic for a given linked symplectic form has good dimensional behavior analogous to that of the classical symplectic Grassmannian.

1. Introduction

Higher-rank Brill-Noether theory studies moduli spaces for pairs $(\mathscr{E}, V)$ on a (smooth, projective) curve C, where $\mathscr{E}$ is a vector bundle of specified rank and degree, and V is a vector space of global sections of $\mathscr{E}$, having specified dimension. The classical case of line bundles is now well understood, with most proofs of the main theorems relying on degeneration techniques. In the higher-rank case, there are a number of partial results, with many of them using the generalization of the Eisenbud-Harris theory of limit linear series to higher-rank vector bundles given in [4]. However, even in the case of rank 2, we have no comprehensive conjectures on the dimensions of the components of the moduli spaces, or when they are nonempty. See [2] for a survey.

One phenomenon observed by Bertram, Feinberg and Mukai in [1] and [5] is that loci of bundles of rank 2 and canonical determinant always have larger than the expected dimension, due to additional symmetries in this case. This was generalized to other special determinants in [6]. In the case of canonical determinant, Bertram, Feinberg and Mukai conjectured that the moduli spaces in question were always nonempty when their modified expected dimension was nonnegative, and this conjecture remains open. In order to use limit linear series techniques to prove existence results in this setting, it is necessary to understand how the symmetries arising from special determinants interact with the expected dimension bounds of the moduli spaces of generalized limit linear series. In the generalized Eisenbud-Harris setting, this is not at all obvious, and our motivation is to prove that we obtain the necessary modified expected dimension bounds, using the alternative construction of limit linear series spaces presented in [8]. We emphasize that such results have immediate implications: [9] proves rather strong existence results towards the Bertram-Feinberg-Mukai conjecture assuming that the desired dimension bounds hold for families of limit linear series on degenerations to chains of elliptic curves.

In the limit linear series construction of [8], spaces of linked Grassmannians are introduced to serve as ambient moduli spaces. Given a base scheme S, integers $r < d$, let $\mathscr{E}_\bullet$ denote a chain of vector bundles $\mathscr{E}_1, \ldots, \mathscr{E}_n$ on S of rank d, together

1

with homomorphisms $f_i : \mathscr{E}_i \to \mathscr{E}_{i+1}$ and $f^i : \mathscr{E}_{i+1} \to \mathscr{E}_i$ satisfying certain natural conditions (recalled in Definition 2.1 below). Then the associated linked Grassmannian $LG(r, \mathscr{E}_\bullet)$ parametrizes tuples of subbundles $\mathscr{F}_i \subseteq \mathscr{E}_i$ of rank r, which are all mapped into one another under the f_i and f^i. These schemes behave like flat degenerations of the classical Grassmannian $G(r, d)$, and indeed according to [3], whenever the f_i and f^i are generically isomorphisms, the linked Grassmannian does in fact yield a flat degeneration of $G(r, d)$. The basic idea of the construction of limit linear series spaces in [8] is that one replaces the Grassmannian used in the construction of linear series spaces on smooth curves with a linked Grassmannian. On the other hand, one may express the modified expected dimension observed by Bertram, Feinberg and Mukai by saying that one replaces the Grassmannian by a symplectic Grassmannian. In order to combine the two, we thus wish to introduce a notation of linked symplectic Grassmannian, and to prove that it has good dimension behavior.

We first introduce a more general notion of a linked alternating Grassmannian, based on a definition of linked alternating form. It turns out that the key idea is to not only consider alternating forms on each of the $\mathscr{E}_i$, but also pairings between $\mathscr{E}_i$ and $\mathscr{E}_j$ for $i \neq j$, satisfying certain natural compatibility conditions. We prove:

Theorem 1.1. *If $LAG(r, \mathscr{E}_\bullet, \langle,\rangle_\bullet) \subseteq LG(r, \mathscr{E}_\bullet)$ is a linked alternating Grassmannian, and $z \in LAG(r, \mathscr{E}_\bullet, \langle,\rangle_\bullet)$ is a smooth point of $LG(r, \mathscr{E}_\bullet)$, then locally at z, we have that $LAG(r, \mathscr{E}_\bullet, \langle,\rangle_\bullet)$ is cut out by $\binom{r}{2}$ equations inside $LG(r, \mathscr{E}_\bullet)$.*

We then define linked symplectic forms to be linked alternating forms satisfying a certain nondegeneracy condition, and thus define linked symplectic Grassmannians to be the corresponding special case of linked alternating Grassmannians. Via an analysis of tangent spaces, we prove:

Theorem 1.2. *If $LSG(r, \mathscr{E}_\bullet, \langle,\rangle_\bullet) \subseteq LG(r, \mathscr{E}_\bullet)$ is a linked symplectic Grassmannian, and S is regular, and $z \in LSG(r, \mathscr{E}_\bullet, \langle,\rangle_\bullet)$ is a smooth point of $LG(r, \mathscr{E}_\bullet)$, then z is also smooth point of $LSG(r, \mathscr{E}_\bullet, \langle,\rangle_\bullet)$, and the latter has codimension $\binom{r}{2}$ in $LG(r, \mathscr{E}_\bullet)$ at z.*

For the statements of the above theorems, note there is an explicit description of the smooth points of linked Grassmannians, recalled in Theorem 2.7 below.

The restriction to smooth points does not cause problems in proving existence results using limit linear series, as smoothing arguments based on the Eisenbud-Harris theory already require restriction to an open subset of the moduli space of limit linear series which is contained in the smooth points of the ambient linked Grassmannian. The main limitation of our result is then that the construction of [8] has thus far only been carried out for reducible curves with two components. We are thus able to conclude that limit linear series arguments are valid for pairs with canonical determinant and degenerations to curves with two components; see Theorem 6.3. However, in order to obtain the same result for arbitrary curves of compact type, as needed for [9], it is necessary to generalize the construction of [8] to arbitrary curves of compact type.

2. Preliminaries

We work throughout over a fixed base scheme S. We being by recalling some ideas from [8] and [7], rephrased in a more convenient manner.

The basic definition is as follows:

Definition 2.1. Let d, n be positive integers. Suppose that $\mathscr{E}_1, \ldots, \mathscr{E}_n$ are vector bundles of rank d on S and we have homomorphisms

$$f_i : \mathscr{E}_i \to \mathscr{E}_{i+1}, \quad f^i : \mathscr{E}_{i+1} \to \mathscr{E}_i$$

for each $i = 1, \ldots, n - 1$. Given $s \in \Gamma(S, \mathscr{O}_S)$, we say that $\mathscr{E}_\bullet = (\mathscr{E}_i, f_i, f^i)_i$ is s-**linked** if the following conditions are satisfied:

(I) For each $i = 1, \ldots, n$,

$$f_i \circ f^i = s \cdot \mathrm{id}, \text{ and } f^i \circ f_i = s \cdot \mathrm{id}.$$

(II) On the fibers of the $\mathscr{E}_i$ at any point with $s = 0$, we have that for each $i = 1, \ldots, n - 1$,

$$\ker f^i = \mathrm{im}\, f_i, \text{ and } \ker f_i = \mathrm{im}\, f^i.$$

(III) On the fibers of the $\mathscr{E}_i$ at any point with $s = 0$, we have that for each $i = 1, \ldots, n - 2$,

$$\mathrm{im}\, f_i \cap \ker f_{i+1} = (0), \text{ and } \mathrm{im}\, f^{i+1} \cap \ker f^i = (0).$$

If $\mathscr{E}_\bullet$ satisfies conditions (I) and (III), we say it is **weakly** s-**linked**.

The s-linkage condition is precisely that required in [8] for the ambient bundles for a linked Grassmannian, which we also recall below. The following notation will be convenient:

Notation 2.2. In the situation of Definition 2.1, with $i < j$ we write

$$f_{i,j} = f_{j-1} \circ \cdots \circ f_i$$

and

$$f^{j,i} = f^i \circ \cdots \circ f^{j-1}.$$

We also write $f_{i,i} = \mathrm{id}, f^{i,i} = \mathrm{id}$.

We have the following basic structure for s-linked bundles:

Lemma 2.3. *Suppose that $\mathscr{E}_\bullet$ is s-linked. Let $r_i = \mathrm{rk}\, f_i$ for $i = 1, \ldots, n - 1$, and by convention set $r_0 = 0$, $r_n = d$. Then locally on S, for $i = 1, \ldots, n$ there exist subbundles $\mathscr{W}_i \subseteq \mathscr{E}_i$ of rank $r_i - r_{i-1}$ such that:*

    i) *For $i = 2, \ldots, n - 1$ we have that*

$$\mathscr{W}_i \cap \mathrm{span}(\ker f_i, \ker f^{i-1}) = (0),$$

    and similarly $\mathscr{W}_1 \cap \ker f_1 = (0), \mathscr{W}_n \cap \ker f^{n-1} = (0)$.

    ii) *For all $j < i$, the restriction of $f_{j,i}$ to $\mathscr{W}_j$ is an isomorphism onto a subbundle of $\mathscr{E}_i$, and for $j > i$ the restriction of $f^{j,i}$ to $\mathscr{W}_j$ is an isomorphism onto a subbundle of $\mathscr{E}_i$.*

    iii) *The natural map*

$$\left(\bigoplus_{j=1}^{i} f_{j,i}(\mathscr{W}_j) \right) \oplus \left(\bigoplus_{j=i+1}^{n} f^{j,i}(\mathscr{W}_j) \right) \to \mathscr{E}_i$$

    is an isomorphism for each i.

The proof is similar to that of Lemma 2.5 of [7] (see also the proof of Lemma A.12 (ii) of [8]), and is omitted.

We next discuss moduli of linked subbundles.

Definition 2.4. Given $\mathscr{E}_\bullet$ weakly s-linked of rank d, and $r < d$, a **linked subbundle** $\mathscr{F}_\bullet \subseteq \mathscr{E}_\bullet$ of rank r consists of a subbundle $\mathscr{F}_i \subseteq \mathscr{E}_i$ for each i such that $f_i \mathscr{F}_i \subseteq \mathscr{F}_{i+1}$, and $f^i \mathscr{F}_{i+1} \subseteq \mathscr{F}_i$ for $i = 1, \ldots, n-1$.

Note that a linked subbundle automatically inherits the structure of a weakly s-linked subbundle, but it is not necessarily the case that a linked subbundle of an s-linked bundle is s-linked.

Definition 2.5. Suppose $\mathscr{E}_\bullet$ is s-linked of rank d, and we are given $r < d$. Then the **linked Grassmannian** $LG(r, \mathscr{E}_\bullet)$ is the scheme representing the functor of linked subbundles of $\mathscr{E}_\bullet$ of rank r.

It is easy to see that $LG(r, \mathscr{E}_\bullet)$ exists, and is in fact a projective scheme over S, as it is cut out as a closed subscheme of the product of classical Grassmannians

$$G(r, \mathscr{E}_1) \times_S \cdots \times_S G(r, \mathscr{E}_n).$$

Also note that for fibers with $s \neq 0$, condition (I) of s-linkage implies that all the f_i and f^i are isomorphisms, so any subbundle $\mathscr{F}_i$ uniquely determines the others. Thus, the corresponding fiber of $LG(r, \mathscr{E}_\bullet)$ is isomorphic to the classical Grassmannian $G(r, d)$. The interesting question is thus what happens at points with $s = 0$.

An important definition is:

Definition 2.6. Given $\mathscr{E}_\bullet$ s-linked on S, a morphism $T \to S$, and a linked subbundle $\mathscr{F}_\bullet \subseteq \mathscr{E}_\bullet|_T$ of rank r, we say that $\mathscr{F}_\bullet$ is an **exact point** of $LG(r, \mathscr{E}_\bullet)$ if on the fibers of the $\mathscr{F}_i$ at any point of T with $s = 0$, we have that for each $i = 1, \ldots, n-1$,

$$\ker f^i = \operatorname{im} f_i, \text{ and } \ker f_i = \operatorname{im} f^i.$$

Equivalently, $\mathscr{F}_\bullet$ is an exact point if it is s-linked. It is not hard to see that the exact points form an open subscheme of $LG(r, \mathscr{E}_\bullet)$. The main results of [8] on linked Grassmannians, illustrating the value of the s-linkage condition, are then the following:

Theorem 2.7. *If $\mathscr{E}_\bullet$ is s-linked of rank d, and we are given $r < d$, the exact points of $LG(r, \mathscr{E}_\bullet)$ are precisely the smooth points of $LG(r, \mathscr{E}_\bullet)$ over S. They have relative dimension $r(d - r)$, and are dense in every fiber.*

These are Lemma A.12, Proposition A.13, and Lemma A.14 of [8].

We thus see that the linked Grassmannian gives degenerations of the classical Grassmannian (in fact, according to the main result of [3] these are flat and Cohen-Macaulay degenerations, but this will not be important for us). The following result, which is contained in the proof of Lemma A.14 of [8], will also be important:

Lemma 2.8. *In the case that S is a point, let $\mathscr{F}_\bullet \subseteq \mathscr{E}_\bullet$ be an exact point of $LG(r, \mathscr{E}_\bullet)$, and let $(\mathscr{W}_i \subseteq \mathscr{F}_i)_i$ be as in Lemma 2.3. Then the tangent space to $LG(r, \mathscr{E}_\bullet)$ at the point corresponding to $\mathscr{F}_\bullet$ is canonically identified with $\bigoplus_i \operatorname{Hom}(\mathscr{W}_i, \mathscr{E}_i/\mathscr{W}_i)$.*

3. Linked alternating forms

We now introduce the definitions of linked bilinear form and linked alternating form which will be central to our analysis.

Definition 3.1. Given a weakly s-linked $\mathscr{E}_\bullet = (\mathscr{E}_i, f_i, f^i)_i$, and $m \in \frac{1}{2}\mathbb{Z}$ between 1 and n, a **linked bilinear form** of **index** m on $\mathscr{E}_\bullet$ is a collection of bilinear pairings for each i, j

$$\langle, \rangle_{i,j} : \mathscr{E}_i \times \mathscr{E}_j \to \mathscr{O}_S$$

satisfying the following compatibility conditions: for all suitable i, j, we have

$$\langle, \rangle_{i,j} \circ (f_{i-1} \times \mathrm{id}) = s^{\epsilon_{i,j}} \langle, \rangle_{i-1,j},$$

$$\langle, \rangle_{i,j} \circ (\mathrm{id} \times f_{j-1}) = s^{\epsilon_{i,j}} \langle, \rangle_{i,j-1},$$

$$\langle, \rangle_{i,j} \circ (f^i \times \mathrm{id}) = s^{\epsilon^{i,j}} \langle, \rangle_{i+1,j}, \quad \text{and}$$

$$\langle, \rangle_{i,j} \circ (\mathrm{id} \times f^j) = s^{\epsilon^{i,j}} \langle, \rangle_{i,j+1},$$

where $\epsilon_{i,j} = \begin{cases} 1 : i+j > 2m \\ 0 : \text{otherwise,} \end{cases}$ and $\epsilon^{i,j} = \begin{cases} 1 : i+j < 2m \\ 0 : \text{otherwise.} \end{cases}$

Note that the data determining a linked bilinear form can be described equivalently as a bilinear form on $\bigoplus_i \mathscr{E}_i$, but it is harder to describe the compatibility conditions in this context. We also observe that if m is an integer, then each fixed-index form $\langle, \rangle_{i,i}$ is induced from $\langle, \rangle_{m,m}$ by setting

$$\langle, \rangle_{i,i} = \langle, \rangle_{m,m} \circ (f_{i,m} \times f_{i,m})$$

for $i < m$, and similarly with $f^{i,m}$ for $i > m$. For further discussion of the motivation for and consequences of the compatibility conditions, see Remark 3.8 below.

The following lemma checks that our compatibility conditions are internally consistent, and will be useful later. Its proof is trivial from the definitions.

Lemma 3.2. *The compatibility conditions imposed in Definition 3.1 satisfy the following consistencies:*

i) *For all suitable i, j, they impose that*

$$\langle, \rangle_{i,j} \circ (f_{i-1} \times \mathrm{id}) \circ (f^{i-1} \times \mathrm{id}) = s \langle, \rangle_{i,j},$$

$$\langle, \rangle_{i,j} \circ (f^i \times \mathrm{id}) \circ (f_i \times \mathrm{id}) = s \langle, \rangle_{i,j},$$

$$\langle, \rangle_{i,j} \circ (\mathrm{id} \times f_{j-1}) \circ (\mathrm{id} \times f^{j-1}) = s \langle, \rangle_{i,j}, \quad \text{and}$$

$$\langle, \rangle_{i,j} \circ (\mathrm{id} \times f^j) \circ (\mathrm{id} \times f_j) = s \langle, \rangle_{i,j}.$$

Formally,

$$\epsilon_{i,j} + \epsilon^{i-1,j} = 1,$$

$$\epsilon_{i+1,j} + \epsilon^{i,j} = 1,$$

$$\epsilon_{i,j} + \epsilon^{i,j-1} = 1, \quad \text{and}$$

$$\epsilon_{i,j+1} + \epsilon^{i,j} = 1.$$

ii) *For all suitable i, j, they impose that*

$$\langle, \rangle_{i,j} \circ (f_{i-1} \times \mathrm{id}) \circ (\mathrm{id} \times f_{j-1}) = \langle, \rangle_{i,j} \circ (\mathrm{id} \times f_{j-1}) \circ (f_{i-1} \times \mathrm{id}),$$

$$\langle, \rangle_{i,j} \circ (f_{i-1} \times \mathrm{id}) \circ (\mathrm{id} \times f^j) = \langle, \rangle_{i,j} \circ (\mathrm{id} \times f^j) \circ (f_{i-1} \times \mathrm{id}),$$

$$\langle, \rangle_{i,j} \circ (f^i \times \mathrm{id}) \circ (\mathrm{id} \times f_{j-1}) = \langle, \rangle_{i,j} \circ (\mathrm{id} \times f_{j-1}) \circ (f^i \times \mathrm{id}), \ and$$

$$\langle, \rangle_{i,j} \circ (f^i \times \mathrm{id}) \circ (\mathrm{id} \times f^j) = \langle, \rangle_{i,j} \circ (\mathrm{id} \times f^j) \circ (f^i \times \mathrm{id}).$$

Formally,

$$\epsilon_{i,j} + \epsilon_{i-1,j} = \epsilon_{i,j} + \epsilon_{i,j-1},$$
$$\epsilon_{i,j} + \epsilon^{i-1,j} = \epsilon^{i,j} + \epsilon_{i,j+1},$$
$$\epsilon^{i,j} + \epsilon_{i+1,j} = \epsilon_{i,j} + \epsilon^{i,j-1}, \ and$$
$$\epsilon^{i,j} + \epsilon^{i+1,j} = \epsilon^{i,j} + \epsilon^{i,j+1}.$$

The alternating condition is then imposed as follows.

Definition 3.3. In the notation of Definition 3.1, a linked bilinear form is a **linked alternating form** if $\langle, \rangle_{i,i}$ is an alternating form on $\mathscr{E}_i$ for all i, and $\langle, \rangle_{i,j} = -\langle, \rangle_{j,i} \circ \mathrm{sw}_{i,j}$ for all $i \neq j$, where $\mathrm{sw}_{i,j} : \mathscr{E}_i \times \mathscr{E}_j \to \mathscr{E}_j \times \mathscr{E}_i$ is the canonical switching map.

This definition is equivalent to requiring that the induced form on $\bigoplus_i \mathscr{E}_i$ be alternating.

Observe that being s-linked or weakly s-linked is preserved by base change. It thus makes sense to define moduli functors of linked bilinear forms and linked alternating forms (and indeed, one can do this without any linkage conditions, if s is given). Moreover, it is clear that these functors have natural module structures, and are representable. Our first result is that for s-linked bundles, the moduli of linked bilinear forms and of linked alternating forms behave just like their classical counterparts.

Proposition 3.4. *Suppose $\mathscr{E}_\bullet = (\mathscr{E}_i, f_i, f^i)_i$ is s-linked, and $m \in \frac{1}{2}\mathbb{Z}$ is between 1 and n. Then the moduli scheme of linked bilinear forms on $\mathscr{E}_\bullet$ of index m is a vector bundle on S of rank d^2, and the moduli scheme of linked alternating forms on $\mathscr{E}_\bullet$ of index m is a vector bundle on S of rank $\binom{d}{2}$.*

Proof. First, choose subbundles $\mathscr{F}_i \subseteq \mathscr{E}_i$ as provided by Lemma 2.3. Clearly, a linked bilinear form on $\mathscr{E}_\bullet$ induces by restriction a collection of bilinear pairings

$$\langle, \rangle'_{i,j} : \mathscr{F}_i \times \mathscr{F}_j \to \mathscr{O}_S,$$

or equivalently a bilinear form on $\bigoplus_i \mathscr{F}_i$, and our claim is that this restriction map induces an isomorphism of functors from linked bilinear forms to bilinear forms on $\bigoplus_i \mathscr{F}_i$. Because $\sum_i \mathrm{rk}\,\mathscr{F}_i = d$, the claim yields the first statement of the proposition.

To prove the claim, suppose we have a collection of $\langle, \rangle'_{i,j}$ as above; we aim to construct an inverse to the restriction map. Because

$$\mathscr{E}_i \cong \left(\bigoplus_{j=1}^{i} f_{j,i}(\mathscr{F}_j) \right) \oplus \left(\bigoplus_{j=i+1}^{n} f^{j,i}(\mathscr{F}_j) \right),$$

in order to define $\langle v_1, v_2 \rangle_{i,j}$, it is enough to do so for v_1 in either $f_{\ell,i} \mathscr{F}_\ell$ with $\ell \leq i$, or $f^{\ell,i} \mathscr{F}_\ell$ with $\ell > i$, and v_2 in either $f_{\ell',j} \mathscr{F}_{\ell'}$ with $\ell' \leq j$, or $f^{\ell',j} \mathscr{F}_{\ell'}$ with

$\ell' > j$. Starting from the necessity of having $\langle , \rangle_{i,j} = \langle , \rangle'_{i,j}$ on $\mathscr{F}_i \times \mathscr{F}_j$, we then see that inductive application of the compatibility conditions of Definition 3.1 uniquely determine $\langle , \rangle_{i,j}$. We have to check that the resulting $\langle , \rangle_{i,j}$ is well defined, and satisfies all the compatibility conditions. This is straightforward to verify case by case, using Lemma 3.2. The claim then follows, as the preceding construction is visibly inverse to the restriction map.

To obtain the second statement of the proposition, it is enough to observe that under the isomorphism of functors constructed above, a linked bilinear form is alternating if and only if the induced form on $\bigoplus_i \mathscr{F}_i$ is alternating. Indeed, this follows from the symmetry of the compatibility conditions together with Lemma 3.2. $\qquad\square$

Proposition 3.4 has immediate consequences for loci of isotropy. The relevant definitions are as follows.

Definition 3.5. If $\mathscr{E}_\bullet$ is weakly s-linked, with a linked bilinear form $\langle , \rangle_\bullet$, we say that $\mathscr{E}_\bullet$ is **isotropic** for $\langle , \rangle_\bullet$ if for all i, j, we have that $\langle , \rangle_{i,j}$ vanishes uniformly.

Definition 3.6. If $\mathscr{E}_\bullet$ is weakly s-linked, with a linked bilinear form $\langle , \rangle_\bullet$, the **locus of isotropy** of $\langle , \rangle_\bullet$ on $\mathscr{E}_\bullet$ is the closed subscheme of S representing the functor of morphisms $T \to S$ such that $\mathscr{E}_\bullet$ is isotropic for $\langle , \rangle_\bullet$ after restriction to T.

The fact that the locus of isotropy is represented by a closed subscheme is clear, as $\mathscr{E}_\bullet$ together with $\langle , \rangle_\bullet$ induces a morphism from S to the moduli scheme of linked bilinear forms on $\mathscr{E}_\bullet$, and the locus of isotropy is the preimage under this morphism of the zero form.

Proposition 3.4 thus implies:

Corollary 3.7. *Suppose $(\mathscr{E}_i, f_i, f^i)_i$ is s-linked, and $(\langle , \rangle_{i,j})_{i,j}$ is a linked bilinear (respectively, linked alternating) form on S. Then the locus of S on which $(\mathscr{E}_i)_i$ is isotropic is locally cut out by d^2 (respectively, $\binom{d}{2}$) equations, and thus if S is locally Noetherian, every component of this locus has codimension at most d^2 (respectively, $\binom{d}{2}$) in S.*

Remark 3.8. We conclude with a discussion of the motivation for Definition 3.1. The idea, at least in the case that $m \in \mathbb{Z}$, is that all of the forms are induced from a single form $\langle , \rangle_{m,m}$, which in our ultimate application will be nondegenerate. In this situation, we cannot avoid having $\langle , \rangle_{i,i}$ be degenerate on $\ker f_i$ for $i < m$ and $\ker f^i$ for $i > m$, and examples show that if we only consider the forms $\langle , \rangle_{i,i}$, we will not obtain the behavior we want. Because the $\langle , \rangle_{i,i}$ are not uniformly zero, there is no way to modify them to make them nondegenerate.

However, if suppose that $S = \operatorname{Spec} A$, with A a DVR, and $s \in A$ a uniformizer, and if we have a nondegenerate form $\langle , \rangle_{m,m}$, then the maps $f_\bullet$ and $f^\bullet$ induce not only forms $\langle , \rangle_{i,i}$, but also pairings $\langle , \rangle_{i,j}$ for all i, j. In the cases of interest to us however, on the special fiber we will have $\operatorname{im} f_{m-1}$ orthogonal to $\operatorname{im} f^m$ in $\mathscr{E}_m$, so if we simply take the induced pairings, we will have $\langle , \rangle_{i,j} = 0$ uniformly on the special fiber if $i < m$ and $j > m$, or vice versa. But this means that considered over all of S, the forms $\langle , \rangle_{i,j}$ are multiples of s, and we can factor out powers of s (of exponent equal to $\min(|m - i|, |m - j|)$) so that the form does not vanish uniformly on the special fiber. In the cases of interest to us, we will actually obtain nondegenerate forms this way when $i + j = 2m$.

For an example of the importance of this additional nondegeneracy, see Example 5.3.

4. Linked symplectic forms

Our next task is to give a suitable notion of nondegeneracy for linked alternating forms, which we will use to define linked symplectic Grassmannians.

Definition 4.1. Suppose that $\mathscr{E}_\bullet = (\mathscr{E}_i, f_i, f^i)_i$ is weakly s-linked, and $\langle,\rangle_\bullet$ is a linked alternating form on $\mathscr{E}_\bullet$. We say that $\langle,\rangle_\bullet$ is a **linked symplectic form** if the following conditions are satisfied:

(I) for all i, j between 1 and n with $i + j = 2m$, we have $\langle,\rangle_{i,j}$ nondegenerate.

(II) if $2m < n + 1$, then on all fibers where $s = 0$, and for all i with $2m - 1 < i \le n$, the degeneracy of $\langle,\rangle_{i,1}$ is equal to $\ker f^{i-1}$.

(III) if $2m > n + 1$, then on all fibers where $s = 0$, and for all i with $1 \le i < 2m - n$, the degeneracy of $\langle,\rangle_{i,n}$ is equal to $\ker f_i$.

Note that in conditions (II) and (III), the compatibility conditions of Definition 3.1 imply that the degeneracy is at least the specified subspaces, so all of the conditions are nondegeneracy conditions, and we obtain an open subset of all linked alternating forms.

The following construction will be used to analyze the tangent space to the linked symplectic Grassmannian.

Definition 4.2. Suppose that $\mathscr{E}_\bullet = (\mathscr{E}_i, f_i, f^i)_i$ is s-linked, and $\langle,\rangle_\bullet$ is a linked alternating form on $\mathscr{E}_\bullet$. Let $\mathscr{F}_\bullet \subseteq \mathscr{E}_\bullet$ be an exact linked subbundle, and suppose that $\mathscr{F}_\bullet$ is isotropic for (the restriction of) $\langle,\rangle_\bullet$. Finally, let $(\mathscr{W}_i \subseteq \mathscr{F}_i)_i$ be as in Lemma 2.3. Given a tuple of homomorphisms $(\varphi_i : \mathscr{W}_i \to \mathscr{E}_i/\mathscr{W}_i)_{1=1,\ldots,n}$ define the associated linked alternating form $\langle,\rangle_\bullet^{\varphi_\bullet}$ on $\mathscr{F}_\bullet$ by applying the following formula on the $\mathscr{W}_i$:

$$\langle,\rangle_{i,j}^{\varphi_\bullet} = \langle,\rangle_{i,j} \circ (\varphi_i \times \mathrm{id}) + \langle,\rangle_{i,j} \circ (\mathrm{id} \times \varphi_j).$$

Note that this is well-defined because $\mathscr{F}_\bullet$ is assumed to be isotropic. Also, recall that by Proposition 3.4, the pairings on the $\mathscr{W}_i$ defined above uniquely determine a linked alternating form $\langle,\rangle_\bullet^{\varphi_\bullet}$ on $\mathscr{F}_\bullet$. We then have the following consequence of the symplectic condition:

Lemma 4.3. *In the situation of Definition 4.2, suppose further that $\langle,\rangle_\bullet$ is a linked symplectic form, and that S is a point. Then the map from $\bigoplus_{i=1}^n \mathrm{Hom}(\mathscr{W}_i, \mathscr{E}_i/\mathscr{W}_i)$ to the space of linked alternating forms on $\mathscr{F}_\bullet$ is surjective.*

For the proof of Lemma 4.3, the following lemma is helpful. The proof is an immediate consequence of the compatibility conditions of Definition 3.3.

Lemma 4.4. *Let $\mathscr{E}_\bullet$ be s-linked, with $\langle,\rangle_\bullet$ a linked bilinear form of index m on $\mathscr{E}_\bullet$. Then:*

i) *given i, j with $i + j > 2m$, and any ℓ between 1 and n with $2m - i \le \ell < j$, we have*
$$\langle,\rangle_{i,j} = \langle,\rangle_{i,\ell} \circ (\mathrm{id} \times f^{j,\ell});$$

ii) *given i, j with $i + j < 2m$, and any ℓ between 1 and n with $j < \ell \le 2m - i$, we have*
$$\langle,\rangle_{i,j} = \langle,\rangle_{i,\ell} \circ (\mathrm{id} \times f_{j,\ell});$$

iii) *given i, j with $i + j > 2m$, and any ℓ between 1 and n with $2m - j \leq \ell < i$,*
   we have
$$\langle,\rangle_{i,j} = \langle,\rangle_{\ell,j} \circ (f^{i,\ell} \times \mathrm{id});$$

iv) *given i, j with $i + j < 2m$, and any ℓ between 1 and n with $i < \ell \leq 2m - j$,*
   we have
$$\langle,\rangle_{i,j} = \langle,\rangle_{\ell,j} \circ (f_{i,\ell} \times \mathrm{id}).$$

Note in particular that if, for instance, $2m - i$ is between 1 and n, then $\langle,\rangle_{i,j}$ is induced from $\langle,\rangle_{i,2m-i}$.

We will also use the following easy lemma from linear algebra:

Lemma 4.5. *Suppose $\langle,\rangle : V \times W \to k$ is a non-degenerate bilinear pairing of k-vector spaces, and we have subspaces $W_1 \subsetneq W_2 \subseteq W$ and $V' \subseteq V$. If $(V')^\perp \cap W_2 \subseteq W_1$ in W, then $V' \cap W_2^\perp \subsetneq V' \cap W_1^\perp$ in V.*

Proof of Lemma 4.3. If we choose bases v_j^i for each $\mathscr{W}_i$, it is clearly enough to prove that for all i, j, p, q, unless $i = j$ and $p = q$ there exists a choice of $\varphi_\bullet$ such that the pairing $\left\langle v_{p'}^{i'}, v_{q'}^{j'} \right\rangle_{i',j'}^{\varphi_\bullet}$ is zero for all i', j', p', q' except $i' = i, j' = j, p' = p, q' = q$, or $i' = j, j' = i, p' = q, q' = p$, and in these last two cases, the pairing is nonzero. Given i, j, p, q, first suppose $|m - i| \leq |m - j|$. Then set $\varphi_{i'} = 0$ for all $i' \neq i$, and set $\varphi_i(v_{p'}^i) = 0$ for all $p' \neq p$. We then wish to show that there exists a choice of $\varphi_i(v_p^i) \in \mathscr{E}_i$ such that $\langle \varphi_i(v_p^i), v_q^j \rangle_{i,j} \neq 0$, but $\left\langle \varphi_i(v_p^i), v_{q'}^{j'} \right\rangle_{i,j'} = 0$ for all other choices of j', q'. Equivalently, if we denote by $\hat{\mathscr{W}}_j \subseteq \mathscr{W}_j$ the span of the $v_{q'}^j$ for $q' \neq q$, we want
$$\varphi_i(v_p^i) \in (\hat{\mathscr{W}}_j)^\perp \cap \left(\cap_{j' \neq j} \mathscr{W}_{j'}^\perp \right),$$
but
$$\varphi_i(v_p^i) \notin \cap_{j'=1}^n \mathscr{W}_{j'}^\perp.$$
Here each orthogonal space should be taken with respect to the appropriate pairing.

Now, if we have $1 \leq 2m - i \leq n$, then according to Lemma 4.4, the above conditions are equivalent to having

$$\varphi_i(v_p^i) \in (f\hat{\mathscr{W}}_j)^\perp \cap \left(\bigcap_{j' \neq j, j' \leq 2m-i} (f_{j',2m-i}\mathscr{W}_{j'})^\perp \right) \cap \left(\bigcap_{j' \neq j, j' > 2m-i} (f^{j',2m-i}\mathscr{W}_{j'})^\perp \right)$$

$$= \left(f(\hat{\mathscr{W}}_j) \oplus \left(\bigoplus_{j' \neq j, j' \leq 2m-i} f_{j',2m-i}\mathscr{W}_{j'} \right) \oplus \left(\bigoplus_{j' \neq j, j' > 2m-i} f^{j',2m-i}\mathscr{W}_{j'} \right) \right)^\perp,$$

where $f = f_{j,2m-i}$ or $f = f^{j,2m-i}$ as appropriate, but

$$\varphi_i(v_p^i) \notin \left(\bigcap_{j'=1}^{2m-i} (f_{j',2m-i}\mathscr{W}_{j'})^\perp \right) \cap \left(\bigcap_{j'=2m-i+1}^{n} (f^{j',2m-i}\mathscr{W}_{j'})^\perp \right)$$

$$= \left(\left(\bigoplus_{j'=1}^{2m-i} f_{j',2m-i}\mathscr{W}_{j'} \right) \oplus \left(\bigoplus_{j'=2m-i+1}^{n} f^{j',2m-i}\mathscr{W}_{j'} \right) \right)^\perp.$$

The sums are direct sums because of Lemma 2.3, and now all the orthogonal complements are relative to $\langle,\rangle_{i,2m-i}$. Again by Lemma 2.3, the two sums give distinct

subspaces of $\mathscr{F}_{2m-i}$, so by the nondegeneracy of $\langle,\rangle_{i,2m-i}$ imposed in the definition of a linked symplectic form, we conclude that a $\varphi_i(v_p^i)$ satisfying the desired conditions exists in this case.

On the other hand, if $2m - i < 1$, we can still apply Lemma 4.4 to conclude that what we want is equivalent to

$$\varphi_i(v_p^i) \in \left(f^{j,1}(\hat{\mathscr{W}}_j) \oplus \left(\bigoplus_{j'\neq j} f_{j',1}\mathscr{W}_{j'} \right) \right)^{\perp},$$

but

$$\varphi_i(v_p^i) \notin \left(\bigoplus_{j'=1}^{n} f_{j',1}\mathscr{W}_{j'} \right)^{\perp},$$

where the orthogonal complements are relative to $\langle,\rangle_{i,1}$. Now, $2m - 1 < i$, so applying Lemma 4.4 again, what we want is equivalent to

$$f^{i,2m-1}\varphi_i(v_p^i) \in \left(f^{j,1}(\hat{\mathscr{W}}_j) \oplus \left(\bigoplus_{j'\neq j} f_{j',1}\mathscr{W}_{j'} \right) \right)^{\perp},$$

but

$$\varphi_i(v_p^i) \notin \left(\bigoplus_{j'=1}^{n} f_{j',1}\mathscr{W}_{j'} \right)^{\perp},$$

where now the orthogonal complements are relative to $\langle,\rangle_{2m-1,1}$. Since this form is by the symplectic condition nondegenerate, we have that there exist vectors in $\mathscr{E}_{2m-1}$ with the desired properties, and it is enough to show that we may further assume they lie in $f^{i,2m-1}(\mathscr{E}_i)$. To show this, by Lemma 4.5 it is enough to show

$$(4.1) \qquad (f^{i,2m-1}(\mathscr{E}_i))^{\perp} \cap \left(\bigoplus_{j'=1}^{n} f_{j',1}\mathscr{W}_{j'} \right) \subseteq f^{j,1}(\hat{\mathscr{W}}_j) \oplus \left(\bigoplus_{j'\neq j} f_{j',1}\mathscr{W}_{j'} \right).$$

We then observe that the hypothesis that the degeneracy of $\langle,\rangle_{i,1}$ on $\mathscr{E}_i$ is equal to $\ker f^i = \ker f^{i,2m-1} = \ker f^{i,1}$ implies that

$$\mathscr{E}_1 = (f^{i,2m-1}(\mathscr{E}_i))^{\perp} \oplus f^{i,1}(\mathscr{E}_i),$$

since the subspaces are of complementary dimension and have trivial intersection. Finally, we use the hypothesis that $|m - i| \leq |m - j|$ together with $2m - i < 1$ to conclude that $i + j > 2m$, and then that $i - m$ and $j - m$ are both nonnegative, so $j \geq i$. Thus, $f^{j,1}(\mathscr{W}_j) \subseteq f^{i,1}(\mathscr{E}_i)$, so we conclude from the direct sum decomposition of $\mathscr{E}_1$ that the left side of (4.1) is equal to $(f^{i,2m-1}(\mathscr{E}_i))^{\perp} \cap \left(\bigoplus_{j'\neq j} f_{j',1}\mathscr{W}_{j'} \right)$, which yields the desired containment.

The cases that $2m - i > n$ and that $|m - j| \leq |m - i|$ proceed in the same fashion, so we conclude the lemma. $\square$

Remark 4.6. One might wonder whether in the definition of a linked symplectic form, using the notation from the proof of Proposition 3.4, it would not be enough to ask that the induced alternating form on $\bigoplus_i \mathscr{F}_i$ be symplectic. While this condition might seem natural, it is not visibly intrinsic, nor does it arise naturally

from the context of limit linear series. We will see in Example 5.4 below that it is not enough to guarantee the behavior we want.

5. Linked symplectic Grassmannians

We can now proceed to define linked alternating Grassmannians and linked symplectic Grassmannians, and we easily conclude our main results.

Definition 5.1. Given $\mathscr{E}_\bullet$ s-linked with a linked alternating form $\langle,\rangle_\bullet$, the **linked alternating Grassmannian** $LAG(r, \mathscr{E}_\bullet, \langle,\rangle_\bullet)$ is the closed subscheme of $LG(r, \mathscr{E}_\bullet)$ parametrizing linked subbundles which are isotropic for (the restriction of) $\langle,\rangle_\bullet$.

Proof of Theorem 1.1. By definition, $LAG(r, \mathscr{E}_\bullet, \langle,\rangle_\bullet)$ is precisely the isotropy locus of the restriction of $\langle,\rangle_\bullet$ to the universal subbundle on $LG(r, \mathscr{E}_\bullet)$. Since the statement is local, we may restrict to the smooth locus of $LG(r, \mathscr{E}_\bullet)$, which according to Theorem 2.7 is precisely the locus of exact points. On this locus, the universal subbundle is s-linked, and we conclude the desired statement from Corollary 3.7. $\square$

Definition 5.2. Given $\mathscr{E}_\bullet$ s-linked with a linked symplectic form $\langle,\rangle_\bullet$, the **linked symplectic Grassmannian** $LSG(r, \mathscr{E}_\bullet, \langle,\rangle_\bullet)$ is the closed subscheme of $LG(r, \mathscr{E}_\bullet)$ parametrizing linked subbundles which are isotropic for (the restriction of) $\langle,\rangle_\bullet$.

Proof of Theorem 1.2. Once again, $LSG(r, \mathscr{E}_\bullet, \langle,\rangle_\bullet)$ is precisely the isotropy locus of the restriction of $\langle,\rangle_\bullet$ to the universal subbundle on $LG(r, \mathscr{E}_\bullet)$, which we recall is the pullback of the zero section under the induced morphism from $LG(r, \mathscr{E}_\bullet)$ to the space of linked alternating forms on the universal subbundle. We may again restrict to the smooth locus of $LG(r, \mathscr{E}_\bullet)$, so that the space of linked alternating forms is by Proposition 3.4 a vector bundle of rank $\binom{r}{2}$, and we may view $LSG(r, \mathscr{E}_\bullet, \langle,\rangle_\bullet)$ as the intersection of two sections inside this bundle. In order to prove the theorem, it is then enough (see for instance Lemma 4.4 of [6]) to see that the tangent spaces to these sections intersect transversely in the fiber over any point of S. We may thus assume that S is a point, and thus the $\mathscr{E}_\bullet$ are simply vector spaces.

At a point of the zero section, the tangent space of our bundle decomposes canonically as a direct sum of the tangent space of $LG(r, \mathscr{E}_\bullet)$ (which is described by Lemma 2.8) and the tangent space to the moduli space of linked alternating forms on the corresponding fixed linked subspace. Since the latter moduli space is a vector space, the tangent space is identified with the space itself. Given a tangent vector to $LG(r, \mathscr{E}_\bullet)$ at a point, our tautological sections yields a tangent vector in the moduli space of linked alternating forms on the corresponding linked subspace, which we may think of as a linked alternating form. One checks from the definitions that if the tangent vector is represented by $(\varphi_i : \mathscr{W}_i \to \mathscr{E}_i/\mathscr{W}_i)_i$ for some choice of $\mathscr{W}_i$ as in Lemma 2.3, the resulting linked alternating form obtained from the tautological section at this point is precisely $\langle,\rangle_\bullet^{\varphi_\bullet}$ as defined in Definition 4.2. Since tangent vectors to the zero section always yield the zero linked alternating form, transversality of the tangent spaces of the two sections follows from the surjectivity of the map $\varphi_\bullet \to \langle,\rangle_\bullet^{\varphi_\bullet}$, given to us by Lemma 4.3. We thus conclude the theorem. $\square$

We conclude with two examples. The first demonstrates the importance of considering pairings between different spaces in defining a linked alternating form, while the second justifies our definition of a linked symplectic form.

Example 5.3. Consider the case $d = 4$, $n = 3$, $r = 2$, and working over a DVR
with uniformizer s. We suppose we have chosen bases of the ambient spaces so
that $f_1 = f_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & s \end{bmatrix}$, and that $f^1 = f^2 = \begin{bmatrix} s & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. The resulting
linked Grassmannian has relative dimension 4, and we want a symplectic linked
Grassmannian to have relative dimension 3.

First suppose we only consider alternating forms on each individual space, com-
patible with the f_i and f^i. Then if we consider linked alternating forms of index 2,
we could set $\langle,\rangle_{2,2} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$ for maximum nondegeneracy. Compatibil-
ity with f_1 and f^2 then forces

$$\langle,\rangle_{1,1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & s^2 \\ 0 & 0 & -s^2 & 0 \end{bmatrix} \text{ and } \langle,\rangle_{3,3} = \begin{bmatrix} 0 & s^2 & 0 & 0 \\ -s^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

Now, an open subset of linked Grassmannian can be written in the form

$$\mathscr{F}_1 = \mathrm{span}((1, a_1, 0, a_2), (0, s^2 b_1, 1, b_2)),$$
$$\mathscr{F}_2 = \mathrm{span}((1, a_1, 0, sa_2), (0, sb_1, 1, b_2)),$$
$$\mathscr{F}_3 = \mathrm{span}((1, a_1, 0, s^2 a_2), (0, b_1, 1, b_2)).$$

Working over the entire DVR, we see that the condition that these subspaces are
isotropic for $\langle,\rangle_{1,1}$, $\langle,\rangle_{2,2}$, and $\langle,\rangle_{3,3}$ is simply that $sb_1 + sa_2 = 0$. Over the generic
point, we get $b_1 + a_2 = 0$, imposing the desired additional condition. However,
at $s = 0$ we see that the subspaces are automatically isotropic, so we get a full
4-dimensional component of the linked Grassmannian.

In order to obtain the desired relative dimension, we must impose the condition
$b_1 + a_2 = 0$ even on the closed fiber. We thus see that it is necessary to consider also
the pairing $\langle,\rangle_{1,3}$, which according to our compatibility conditions will be given by
$\langle,\rangle_{1,3} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$. The requirement that $\mathscr{F}_1$ be orthogonal to $\mathscr{F}_3$ under
$\langle,\rangle_{1,3}$ then yields the desired condition $b_1 + a_2 = 0$.

Example 5.4. Consider the case $d = 4$, $n = 2$, $r = 2$, working over a base field.
We set $s = 0$, and for a given basis, consider maps of the form $f_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$,
and $f^1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. As before, we obtain a linked Grassmannian of dimension
4, and would want a linked symplectic Grassmannian to have dimension 3.

We obtain a linked alternating form of index 2 by setting $\langle,\rangle_{2,2} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$,

and letting the other pairings be induced from $\langle,\rangle_{2,2}$ as determined by the compatibility conditions. If we set $\mathscr{F}_1 = \mathrm{span}((1,0,0,0),(0,1,0,0))$ and $\mathscr{F}_2 = \mathrm{span}((0,0,1,0),(0,0,0,1))$ then one checks that this induces a symplectic form on $\mathscr{F}_1 \oplus \mathscr{F}_2$. It does not satisfy our conditions for a linked symplectic form, because $\langle,\rangle_{1,1}$ is the zero form, so its degeneracy is strictly larger than $\ker f_1$. Correspondingly, we see that the associated linked alternating Grassmannian is not pure of dimension 3; indeed, it contains the component of the linked Grassmannian on which $V_2 = f_1(V_1)$.

6. Degenerations and rank-2 Brill-Noether loci

We sketch how the symplectic linked Grassmannian may be used to strengthen the limit linear series techniques of [4] to apply also in the case of rank 2 and canonical determinant, at least in the case that the degenerate curve has two components. We defer a more comprehensive exposition until the theory has been generalized to arbitrary curves of compact type. We do, however, state a precise theorem, in terms of the theory of [4].

We assume throughout that X_0 is a reducible projective curve of genus g obtained as $Y \cup Z$, where Y and Z are smooth of genus g_Y and g_Z respectively, and $Y \cap Z = \{P\}$ is an ordinary node. We first recall the definition of limit linear series in this context from [4].

Definition 6.1. Given k, r, d positive integers, a **limit linear series** of dimension k, rank r, and degree d on X_0 consists of a tuple $(E_Y, V_Y, E_Z, V_Z, \varphi_P)$, where:

  i) E_Y, E_Z are vector bundles of rank r on Y and Z respectively;
  ii) V_Y and V_Z are k-dimensional spaces of global sections of E_Y and E_Z respectively;
  iii) φ_P is an isomorphism of the projectivizations of the fibers of E_Y and E_Z at P,

such that there exist

  iv) an integer $a > 0$;
  v) bases $s_1^Y, \ldots, s_k^Y$ of V_Y and $s_1^Z, \ldots, s_k^Z$ of V_Z,

satisfying the following conditions:

  a) $\deg E_Y + \deg E_Z = d + a$;
  b) the orders of vanishing a_i^Y and a_i^Z of s_i^Y and s_i^Z at P satisfy
  $$a_i^Y + a_i^Z \geq a$$
  for all i;
  c) s_i^Y glues to s_i^Z under φ_P for all;
  d) global sections of $E_Y(-aP)$ and $E_Z(-aP)$ are completely determined by their value in the fiber at P.

We then define limit linear series of canonical determinant as follows.

Definition 6.2. Given $k > 0$, let $(E_Y, V_Y, E_Z, V_Z, \varphi_P)$ be a limit linear series of rank 2, degree $2g - 2$, and dimension k on X_0. We say that $(E_Y, V_Y, E_Z, V_Z, \varphi_P)$ has **canonical determinant** if $\det E_Y \cong \omega_Y((d_Y - 2g_Y + 2)P)$, and $\det E_Z \cong \omega_Z((d_Z - 2g_Z + 2)P)$, where $d_Y := \deg E_Y$ and $d_Z := \deg E_Z$.

Our theorem is then the following.

Theorem 6.3. *Given g, k set $\rho_\omega = 3g-3-\binom{k+1}{2}$. Suppose that $(E_Y, V_Y, E_Z, V_Z, \varphi_P)$ is a limit linear series of canonical determinant and dimension k on X_0 such that the inequalities of Definition 6.1 b) are all equalities. Suppose further that the space of such limit linear series on X_0 has dimension ρ_ω at $(E_Y, V_Y, E_Z, V_Z, \varphi_P)$. Then a general smooth curve of genus g has a vector bundle of rank 2 and canonical determinant with at least k linearly independent global sections.*

This theorem can be sharpened in a straightforward way to include stability conditions; for these, we refer the reader to [9].

We now sketch the proof of the theorem. Just as in [8], we define higher-rank limit linear series in terms of a chain of vector bundles E_i on X_0 related by twisting up and down at the nodes P, with maps between them given by inclusion on Y and zero on Z or vice versa. A limit linear series of dimension k then requires such a chain E_i, together with k-dimensional spaces V_i of global sections of E_i, each mapping into one another under the given maps. In a smoothing family of X_0, the definition is the same except that the E_i are related by twisting by Y and Z, which are now divisors on the total space. We prove representability as in Theorem 5.3 of [8], with moduli stacks of vector bundles in place of Picard schemes. Following Proposition 6.6 of [8], we see that the forgetful map to the extremal pairs of E_i and V_i yields a generalized limit linear series in the sense of Definition 6.1, with the possible exception of the gluing condition in part c). However, generalizing the case of refined Eisenbud-Harris limit series, this forgetful map gives an isomorphism above the open locus for which the inequalities of part b) are satisfied with equality. Finally, on this locus, the corresponding points of the ambient linked Grassmannian are all exact.

In the special case of rank 2 and canonical determinant, one derivation of the modified expected dimension for smooth curves is by constructing the moduli space as follows. Let $\mathcal{M}_{2,\omega}(X)$ be the moduli stack of vector bundles of rank 2 and fixed canonical determinant on X; this is smooth of dimension $3g - 3$. Let $\tilde{\mathscr{E}}$ be the universal bundle on $\mathcal{M}_{2,\omega}(X) \times X$, and let D be a sufficently ample effective divisor on X (technically, we must cover $\mathcal{M}_{2,\omega}(X)$ by a nested increasing sequence of open quasicompact substacks, and carry out this construction on each, letting D grow). Let D' be the pullback of D to $\mathcal{M}_{2,\omega}(X) \times X$. Then $p_{1*}\tilde{\mathscr{E}}(D')$ is a vector bundle of rank

$$\deg \tilde{\mathscr{E}} + \operatorname{rk} \tilde{\mathscr{E}} \deg D + \operatorname{rk} \tilde{\mathscr{E}}(1 - g) = 2g - 2 + 2\deg D + 2 - 2g = 2\deg D.$$

Let $G := G(k, p_{1*}\tilde{\mathscr{E}}(D'))$ be the relative Grassmannian on $\mathcal{M}_{2,\omega}(X)$; our moduli space is cut out by the closed condition of subspaces lying in $p_{1*}\tilde{\mathscr{E}}$. We express this condition in terms of the bundle $p_{1*}(\tilde{\mathscr{E}}(D')/\tilde{\mathscr{E}}(-D'))$, which has rank $4\deg D$. We see that because D was chosen to be large, $p_{1*}\tilde{\mathscr{E}}(D')$ is naturally a subbundle, as is $p_{1*}(\tilde{\mathscr{E}}/\tilde{\mathscr{E}}(-D'))$, which also has rank $2\deg D$. Then the inclusion of the universal subbundle on G, together with the pullback from $\mathcal{M}_{2,\omega}(X)$ of $p_{1*}(\tilde{\mathscr{E}}/\tilde{\mathscr{E}}(-D'))$, induces a morphism

$$G \to G(k, p_{1*}(\tilde{\mathscr{E}}(D')/\tilde{\mathscr{E}}(-D')) \times_{\mathcal{M}_{2,\omega}(X)} G(2\deg D, p_{1*}(\tilde{\mathscr{E}}(D')/\tilde{\mathscr{E}}(-D')),$$

and our desired moduli space is precisely the preimage in G of the incidence correspondence in the product.

We now make use of the canonical determinant hypothesis to observe that by choosing local representatives, using the isomorphism $\bigwedge^2 \tilde{\mathscr{E}} \cong p_2^* \omega_X$, and summing residues over points of D, we obtain a symplectic form on $p_{1*}(\tilde{\mathscr{E}}(D')/\tilde{\mathscr{E}}(-D'))$, Moreover, both $p_{1*}\tilde{\mathscr{E}}(D')$ and $p_{1*}(\tilde{\mathscr{E}}/\tilde{\mathscr{E}}(-D'))$ are isotropic for this form, with the former following from the residue theorem, and the latter from the lack of poles. Thus, our induced map in fact has its image in a product of symplectic Grassmannians, and the incidence correspondence has smaller codimension, so we obtain the modified dimension bound for our moduli space cut out in G.

Moving to the limit case, we need to see that the canonical determinant hypothesis gives us (at least locally on $\mathcal{M}_{2,\omega}(X)$) a linked symplectic form on the chain $p_{1*}(\tilde{\mathscr{E}}_i(D')/\tilde{\mathscr{E}}_i(-D'))$, allowing us to extend the above construction. We assume we have a family X/B with smooth generic fiber, and X_0 as above. Working locally, we may assume that each $\tilde{\mathscr{E}}_i$ has a single fixed multidegree on reducible fibers. First suppose that for some m, the determinant of $\tilde{\mathscr{E}}_m$ is isomorphic to $\omega_{X/B}$, the relative dualizing sheaf. In this case, we construct a linked symplectic form of index m by making use of this isomorphism and our given maps between the $\tilde{\mathscr{E}}_i$ to induce maps

$$\tilde{\mathscr{E}}_i \otimes \tilde{\mathscr{E}}_j \to \omega_{X/B}$$

for all i, j, and using a fixed choice of isomorphism $\mathscr{O}_X(Y+Z) \cong \mathscr{O}_X$ to "factor out" any vanishing along X_0. This factoring out will give us nondegeneracy whenever $i + j = 2m$, and we see that we get a linked symplectic form. On the other hand, if no $\tilde{\mathscr{E}}_i$ has determinant $\omega_{X/B}$, then for some i we have $\det \tilde{\mathscr{E}}_i \cong \omega_{X/B}(Y)$. Then we have an induced surjection

$$\tilde{\mathscr{E}}_{i-1} \otimes \tilde{\mathscr{E}}_i \twoheadrightarrow \omega_{X/B},$$

and if we set $m = i - \frac{1}{2}$, we get an induced linked symplectic form as above.

Now we can put together the canonical determinant construction and the limit linear series construction by replacing the symplectic Grassmannians in the above argument by linked symplectic Grassmannians, and checking that in this case we still define the same functor as before. However, because the linked symplectic Grassmannian is by Theorem 1.2 smooth of dimension equal to the usual symplectic Grassmannian (at least, on the open locus of exact points), the dimension count in the limit linear series case goes through exactly as in the case of smooth curves, and we obtain the desired lower bound on dimension. Because the construction goes through for smoothing families, we obtain the smoothing statement Theorem 6.3 just as in the original Eisenbud-Harris theory.

Note that the fact that moduli spaces of vector bundles are no longer proper is irrelevant to us, as we are concerned only with smoothing statements.

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